(SYSTEMS OF PARTICLES & ROTATIONAL MOTION)

PARTICLE :- A particle is defined as an object whose mass is finite but whose size and internal structure can be neglected.

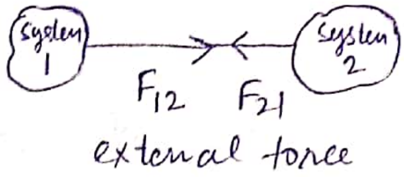
System :- "A system is a collection of a very large number of particles which mutually interact with one another."

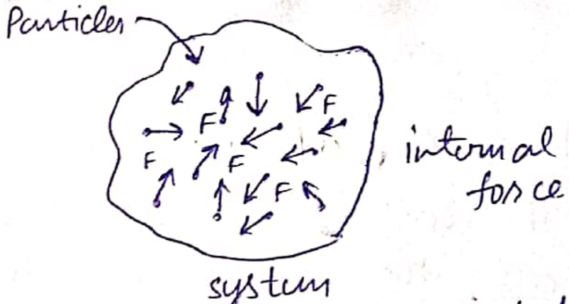
A body of finite size can be considered as a system because it is composed of a large no. of particles interacting with one another.

Internal force :- The mutual forces exerted by the particles of a system on one another are called internal forces.

These forces are responsible for holding together the particles as a single object.

External forces :- The outside force exerted on an object by any external angency is called external force. Such a force changes the velocity of an object





ROTATIONAL MOTION :- When a body of mass m rotates / spining about a fixed axis.

OR

Rotation motion is defined as the motion of a rigid body which takes place in such a way that all of its particles move in circle about an axis with a common angular velocity also the rotation of a particle about a fixed point in space.

Rigid Body ÷ A body is said to be rigid if it does not undergo any change in its size and shape however large the external force may be acting on it

OR

A rigid body is a body that can rotate with all the parts locked together and without any change in its shape.

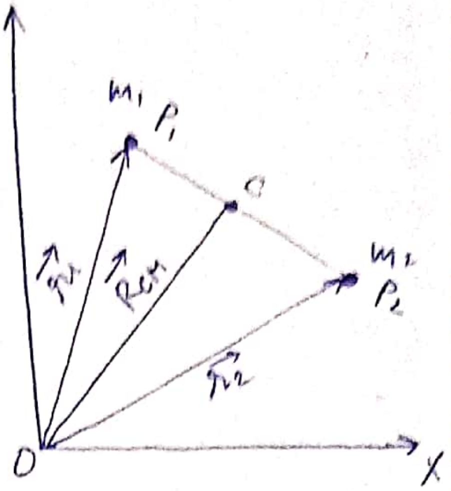
Centre of mass of a system ÷
The centre of mass of a body (system) is a point where the total mass of the body is supposed to be concentrated for describing its translatory motion. and all external forces were applied at that point.
If a single force acts on a body and the line of action of the force passes through the centre of mass the body will have only linear acceleration and no angular acceleration.

Centre of mass vs Centre of gravity ÷
The centre of mass of a body is a point where all the mass of that body may be assumed to be concentrated for describing its translation motion.
On the other hand; "The centre of gravity is a point at which the resultant of the gravitational force on all the particles of the body acts."

In a uniform gravitational field, the centre of gravity coincides with the centre of mass. But at higher altitude the centre of gravity lies a little below its centre of mass because the gravitational force decreases with altitude.

CENTRE OF MASS OF A TWO-PARTICLE SYSTEM :-

Consider a system of two particles P_1 and P_2 of masses m_1 and m_2. Let $\vec{r_1}$ and $\vec{r_2}$ be their position vector with respect to the origin O



The position vector $\vec{R}_{CM}$ of the centre of mass C of the two-particles system is given by.

$$\boxed{\vec{R}_{CM} = \frac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1 + m_2}}$$

NOTE - If $m_1 = m_2 = m$

$$\vec{R}_{CM} = \frac{m\vec{r_1} + m\vec{r_2}}{m + m}$$

$$\vec{R}_{CM} = \frac{\cancel{m}(\vec{r_1} + \vec{r_2})}{2\cancel{m}}$$

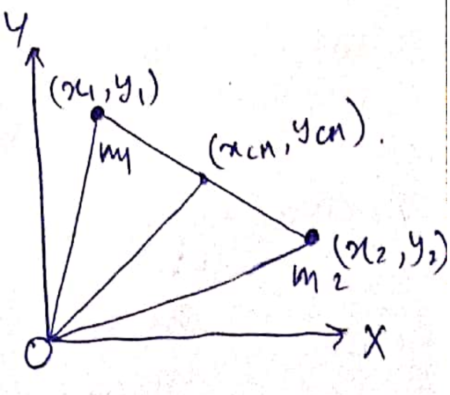
$$\boxed{\vec{R}_{CM} = \frac{\vec{r_1} + \vec{r_2}}{2}}$$

Thus the centre of mass of two equal masses lies exactly at the centre of the line joining the two masses.

- If (x_1, y_1) and (x_2, y_2) are the coordinates of the locations of the two particles, the coordinates of their centre of mass are given by.

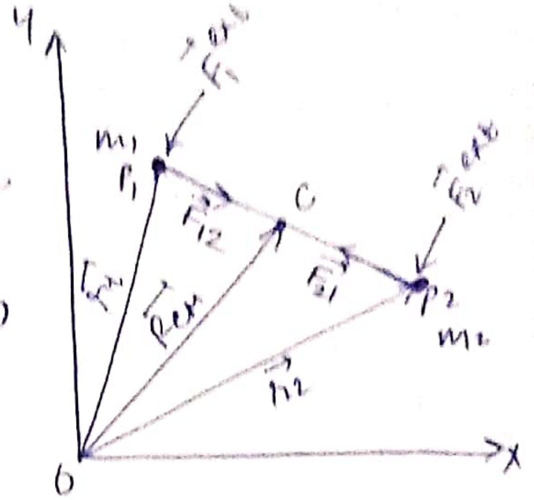
$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$y_{CM} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$$



Derivation of centre of mass of a two particle system from ab-initio :

Consider a system of two particles P_1 and P_2 of masses m_1 and m_2. Let $\vec{r_1}$ and $\vec{r_2}$ be their position vectors at any instant t w.r.t the origin O



The velocity and acceleration vectors of the two particles are

$$\vec{v_1} = \frac{d\vec{r_1}}{dt} \quad , \quad \vec{v_2} = \frac{d\vec{r_2}}{dt}$$

$$\vec{a_1} = \frac{d\vec{v_1}}{dt} = \frac{d^2\vec{r_1}}{dt^2} \quad , \quad \vec{a_2} = \frac{d\vec{v_2}}{dt} = \frac{d^2\vec{r_2}}{dt^2} \quad \text{—— (1)}$$

Total force $\vec{F_1}$ acting on particle P_1 is the sum of the internal force $\vec{F_{12}}$ and due to P_2 and external force $\vec{F_1}^{ext}$ on it. Thus

$$\vec{F_1} = \vec{F_{12}} + \vec{F_1}^{net}$$

Similarly total force on particle P_2 is

$$\vec{F_2} = \vec{F_{21}} + \vec{F_2}^{net}$$

↑ Total force ↑ internal force ↑ external force

According to Newton's second law of motion

$$m_1\vec{a_1} = \vec{F_1} = \vec{F_{12}} + \vec{F_1}^{net} \quad \text{—— (2)}$$

$$m_2\vec{a_2} = \vec{F_2} = \vec{F_{21}} + \vec{F_2}^{net} \quad \text{—— (3)}$$

adding eq^n (2) and (3) we get net force

$$m_1 \vec{a_1} + m_2 \vec{a_2} = \vec{F_{12}} + \vec{F_{21}} + \vec{F_1}^{ext} + \vec{F_2}^{ext}$$

According to newton's third law of motion

$$\vec{F_{12}} = -\vec{F_{21}}$$

$$m_1 \vec{a_1} + m_2 \vec{a_2} = \cancel{\vec{F_{12}}} - \cancel{\vec{F_{12}}} + \vec{F_1}^{ext} + \vec{F_2}^{ext}$$

$$m_1 \vec{a_1} + m_2 \vec{a_2} = \vec{F_1}^{ext} + \vec{F_2}^{ext} \quad \text{——} \quad (4)$$

Now let $\vec{F}$ be the total external force on the both particles

$$\vec{F} = \vec{F_1}^{ext} + \vec{F_2}^{ext}$$

eq^n (4) becomes

$$m_1 \vec{a_1} + m_2 \vec{a_2} = \vec{F} \quad \text{——} \quad (5)$$

Suppose the total mass of the two particles is M. Then $M = m_1 + m_2$

Let the total external force $\vec{F}$ produces acceleration $\vec{a_{CM}}$ in the mass M.

$$\vec{F} = M\vec{a_{cm}} \quad \text{——} \quad (6)$$

From eq^n (5) and (6).

$$M\vec{a_{cm}} = m_1 \vec{a_1} + m_2 \vec{a_2}$$

Using eq^n (i) we get

$$M\vec{a_{cm}} = m_1 \frac{d^2\vec{r_1}}{dt^2} + m_2 \frac{d^2\vec{r_2}}{dt^2}$$

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$$\vec{M}\vec{a}_{CM} = \frac{d^2}{dt^2}(m_1\vec{r}_1 + m_2\vec{r}_2)$$

and $$\vec{a}_{CM} = \frac{d^2 R_{CM}}{dt^2}$$ (acceleration along the centre of mass)

$$\vec{M}\frac{d^2 R_{CM}}{dt^2} = \frac{d^2}{dt^2}(m_1\vec{r}_1 + m_2\vec{r}_2)$$

$$\frac{d^2 R_{CM}}{dt^2} = \frac{d^2}{dt^2}\left(\frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M}\right)$$

But $M = m_1 + m_2$

$$\frac{d^2 R_{CM}}{dt^2} = \frac{d^2}{dt^2}\left(\frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}\right)$$

By comparision

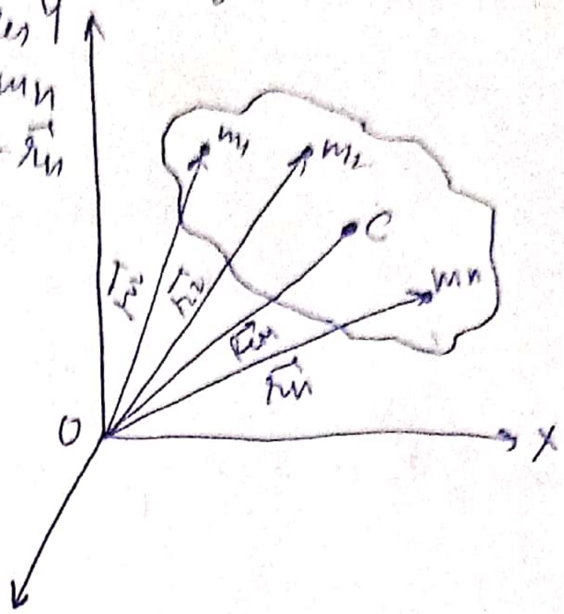
$$\boxed{\vec{R}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}}$$

$\vec{R}_{CM}$ = position vector of the centre of mass.

and we need to consider only the external forces acting on the system to study the motion of centre of mass of any system.

Centre of mass of n-particle system :-

Consider a system of n particles having masses $m_1, m_2, m_3 \ldots m_n$ and position vector $\vec{r_1}, \vec{r_2}, \vec{r_3} \ldots \vec{r_n}$ relative to the origin O.



The total mass of the system

$$M = m_1 + m_2 + m_3 \ldots\ldots + m_n.$$

The position vector $\vec{R_{CM}}$ of the centre of mass C can be obtained by,

$$\vec{R_{CM}} = \frac{m_1\vec{r_1} + m_2\vec{r_2} + m_3\vec{r_3} + \ldots\ldots m_n\vec{r_n}}{m_1 + m_2 + m_3 \ldots\ldots m_n}$$

$$\vec{R_{CM}} = \frac{\sum_{i=1}^{\infty} m_i\vec{r_i}}{\sum_{i=1}^{\infty} m_i}$$

But $\sum_{i=1}^{\infty} m_i = M$

$$\boxed{\vec{R_{CM}} = \frac{\sum_{i=1}^{\infty} m_i\vec{r_i}}{M}}$$

$\vec{R_{CM}}$ is the average of the position vectors of all the particles of the system, the contribution of each particle being proportional to its mass.

NOTE :- In case of a body with a continuous mass distribution. ⇒

The summation will take form

$$\sum_{i=1}^{\infty} m_i r_i = \int \vec{r} dm$$

$$\vec{R}_{CM} = \frac{\sum_{i=1}^{\infty} m_i \vec{r}_i}{M}$$

$$\boxed{\vec{R}_{CM} = \frac{1}{M} \int \vec{r} \cdot dm}$$

The coordinates (x_{CM}, y_{CM}, z_{CM}) of the centre of mass of a body of mass M is given as

$$x_{CM} = \frac{1}{M} \int x \, dm$$

$$y_{CM} = \frac{1}{M} \int y \, dm$$

$$z_{CM} = \frac{1}{M} \int z \, dm$$

• If centre of mass is taken as the origin of our coordinate system. then

$$\vec{R}_{CM}(x, y, z) = 0$$

$$\boxed{\int \vec{r} dm / M = 0} \qquad \left[\vec{R}_{CM} = \frac{1}{M} \int \vec{r} dm \right]$$

$$\boxed{\int \vec{r} dm = 0}$$

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$$\boxed{\int x\,dm = \int y\,dm = \int z\,dm = 0}$$

Cartesian co-ordinates of the centre of mass:-

If x_{CM}, y_{CM}, and z_{CM} are the cartesian co-ordinates of the centre of mass of n-particle system then

$$x_{CM} = \frac{m_1x_1 + m_2x_2 + \cdots m_nx_n}{m_1 + m_2 + m_3 \cdots m_n} = \frac{\sum_{i=1}^{\infty} m_i x_i}{M}$$

Similary

$$y_{CM} = \frac{m_1y_1 + m_2y_2 + \cdots m_ny_n}{m_1 + m_2 + m_3 \cdots m_n} = \frac{\sum_{i=1}^{\infty} m_i y_i}{M.}$$

and

$$z_{CM} = \frac{m_1z_1 + m_2z_2 + \cdots m_nz_n}{m_1 + m_2 + m_3 \cdots m_n} = \frac{\sum_{i=1}^{\infty} m_i z_i}{M}$$

Equation of motion for the centre of mass:-

Let $\vec{F_1}, \vec{F_2}, \vec{F_3} \cdots \vec{F_n}$ be the external forces acting on the particles of masses $m_1, m_2, m_3 \cdots m_n$ respectively.

Let $\vec{F}_{total}$ be the vector sum of all these external forces acting on the system. If $\vec{a}_{CM}$ is the acceleration of the centre of mass of the system then the motion of the centre of mass is given by equation

$$\vec{F}_{total} = \vec{F_1} + \vec{F_2} + \vec{F_3} \cdots \vec{F_n} = M\vec{a}_{CM}$$

But $\boxed{\vec{F}_{total} = M\vec{a}_{CM}}$ where $\vec{a}_{CM} = \frac{d^2\vec{R}_{CM}}{dt^2}$

Thus the centre of mass of the system moves as if the entire mass of the system is concentrated at this point and the total external force acts on this point. The internal forces between the particles cancel out in pairs accordance with Newton's third law of motion.

Show that velocity of centre of mass is constant in the absence of external force ÷

Suppose an external force $\vec{F}_{total}$ acts on a system of mass M and produces an acceleration $\vec{a}_{CM}$ in its centre of mass then

$$\vec{F}_{total} = M\vec{a}_{CM}$$

In the absence of external force

$$\vec{F}_{total} = 0 \quad \text{so}$$

$$M\vec{a}_{CM} = 0 \qquad \left[v_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2} \right]$$

$$\vec{a}_{CM} = 0$$

$$\left[\vec{V}_{CM} = \frac{d\vec{R}_{cm}}{dt}, \right]$$

But $\vec{a}_{CM} = \dfrac{d\vec{V}_{CM}}{dt}$ $\quad [\text{or } R_{CM} = V_{CM} \cdot dt]$

$$\frac{d\vec{V}_{cm}}{dt} = 0$$

"As the derivative of a constant is zero."

$$\boxed{\vec{V}_{CM} = \text{constant}}$$

$\vec{V}_{CM}$ is the velocity of the centre of mass

Hence in the absence of any external force, the centre of mass of system moves with a uniform velocity. This is Newton's first law of motion. The position vector of centre of mass at any instant t is given by $\boxed{\vec{R}_{CM}(t) = \vec{R}_{CM}(0) + \vec{V}_{CM} \cdot t}$

Show that total linear momentum of a system of particles is conserved in the absence of any external force : (Conservation of Momentum).

consider a system of n particles of masses $m_1, m_2, m_3 \ldots m_n$. Suppose the force $\vec{F_1}, \vec{F_2}, \vec{F_3} \ldots \vec{F_n}$ exerted on them produces acceleration $\vec{a_1}, \vec{a_2}, \vec{a_3} \ldots \vec{a_n}$ respectively.

$$\vec{F}_{total} = \vec{F_1} + \vec{F_2} + \vec{F_3} \ldots \vec{F_n}$$

In the absence of external force.

$$\vec{F}_{total} = 0$$

$$\vec{F_1} + \vec{F_2} + \vec{F_3} \ldots \vec{F_n} = 0$$

$$m_1\vec{a_1} + m_2\vec{a_2} + m_3\vec{a_3} \ldots m_n\vec{a_n} = 0$$

$$m_1\frac{d\vec{v_1}}{dt} + m_2\frac{d\vec{v_2}}{dt} + m_3\frac{d\vec{v_3}}{dt} \ldots m_n\frac{d\vec{v_n}}{dt} = 0$$

$$\frac{d}{dt}\Big(\underset{\vec{P_1}}{\underset{\downarrow}{m_1\vec{v_1}}} + \underset{\vec{P_2}}{\underset{\downarrow}{m_2\vec{v_2}}} + \underset{\vec{P_3}}{\underset{\downarrow}{m_3\vec{v_3}}} - \ldots + \underset{\vec{P_n}}{\underset{\downarrow}{m_n\vec{v_n}}}\Big) = 0$$

$$\frac{d}{dt}\left(\vec{P_1} + \vec{P_2} + \vec{P_3} + \ldots + \vec{P_n}\right) = 0$$

$$\vec{P_1} + \vec{P_2} + \vec{P_3} + \ldots + \vec{P_n} = \text{constant}$$

$$\boxed{\vec{P} = \vec{P_1} + \vec{P_2} + \vec{P_3} + \ldots + \vec{P_n} = \text{constant}}$$

where $\vec{P}$ is total linear momentum of the system. Hence if no net external force acts on a system, the total linear momentum of the system is conserved. This is the law of conservation of momentum.

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Show that the linear momentum (total) of the system is equal to the product of the total mass of the system and the velocity of its centre of mass?

Let the position vector of the centre of mass of n - particles is given by

$$\vec{R}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 \cdots + m_n\vec{r}_n}{m_1 + m_2 + m_3 + \cdots + m_n}$$

$$\vec{R}_{CM} = \frac{1}{M}\left(m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 \cdots + m_n\vec{r}_n\right)$$

Differentiating both sides w.r.t time t we get

$$\frac{d\vec{R}_{CM}}{dt} = \frac{1}{M}\frac{d}{dt}\left(m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots + m_n\vec{r}_n\right)$$

$$\frac{d\vec{R}_{CM}}{dt} = \frac{1}{M}\left[m_1\underbrace{\frac{d\vec{r}_1}{dt}}_{\vec{v}_1} + m_2\underbrace{\frac{d\vec{r}_2}{dt}}_{\vec{v}_2} + m_3\underbrace{\frac{d\vec{r}_3}{dt}}_{\vec{v}_3} + \cdots m_n\underbrace{\frac{d\vec{r}_n}{dt}}_{\vec{v}_n}\right]$$

$$\frac{d\vec{R}_{CM}}{dt} = \frac{1}{M}\left[\underbrace{m_1\vec{v}_1}_{\vec{p}_1} + \underbrace{m_2\vec{v}_2}_{\vec{p}_2} + \underbrace{m_3\vec{v}_3}_{\vec{p}_3} + \cdots + \underbrace{m_n\vec{v}_n}_{\vec{p}_n}\right]$$

$$\frac{d\vec{R}_{CM}}{dt} = \frac{1}{M}\left(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \cdots + \vec{p}_n\right)$$

Now $\dfrac{d\vec{R}_{CM}}{dt} = \vec{v}_{CM}$ (Velocity of centre of mass)

$$\vec{v}_{CM} = \frac{1}{M}(\vec{P}_1 + \vec{P}_2 + \vec{P}_3 - - - + \vec{P}_n)$$

and $\vec{P}_1 + \vec{P}_2 + \vec{P}_3 + \cdots + \vec{P}_n = \vec{P}$ (Total linear momentum)

$$\vec{V}_{CM} = \frac{1}{M} \cdot \vec{P}$$

$$\boxed{\vec{P} = M\vec{V}_{CM}}$$

This eqn shows that the total linear momentum of a system is equal to the product of the total mass of the system (M) and the velocity of its centre of mass.